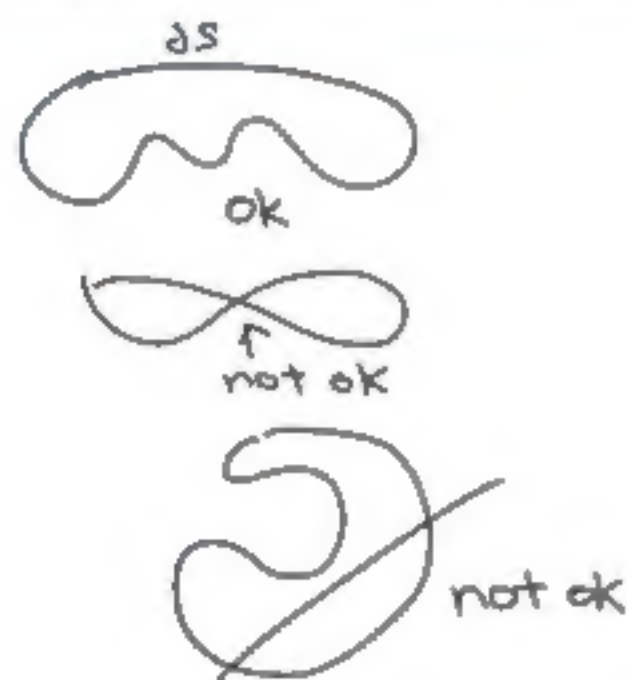


IDEA: Generalize Green's theorem to surfaces which are not flat...

Prop (Stokes's theorem):

suppose S is a piecewise smooth surface with piecewise-smooth boundary curve, which is closed and has only one component. If a vector field with continuous partial derivatives on S , then $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$



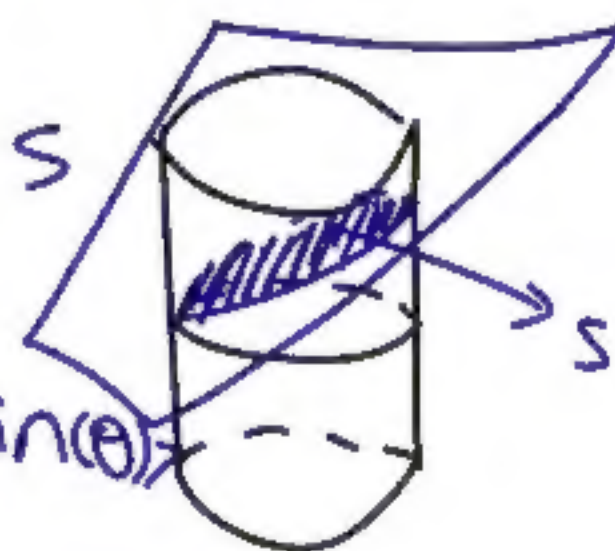
Ex. Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C the curve of intersections of plane $y+z=2$ and cylinder $x^2+y^2=1$

Sol: We need $C = \partial S$ for some surface S

A good choice:

$$\vec{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), 2 - r \sin(\theta) \rangle$$

$$\text{on } (r, \theta) \in [0, 1] \times [0, 2\pi]$$



By Stokes's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \iint_D \text{curl}(\vec{F})(\vec{r}(r, \theta)) \cdot (\vec{r}_u \times \vec{r}_v) dA$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y \rangle$$

$$\text{curl}(\vec{F})(s(r, \theta)) = \langle 0, 0, 1 + 2r \sin(\theta) \rangle$$

$$\vec{s}_r = \langle \cos(\theta), \sin(\theta), -\sin(\theta) \rangle$$

$$\vec{s}_\theta = \langle -r \sin(\theta), r \cos(\theta), -r \cos(\theta) \rangle$$

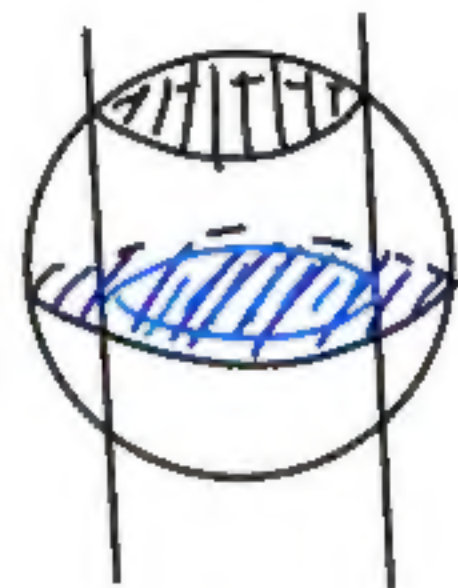
$$\begin{aligned} \therefore \vec{s}_r \times \vec{s}_\theta &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -\sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) & -r \cos(\theta) \end{bmatrix} \\ &= r \langle 0, 1, 1 \rangle \quad \text{has correct orientation} \\ &\quad \text{for counterclockwise} \\ &\quad \text{from above.} \end{aligned}$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \iint_D \langle 0, 0, 1 + 2r \sin(\theta) \rangle \cdot r \langle 0, 1, 1 \rangle dA \\ &= \int_{r=0}^1 \int_{\theta=0}^{2\pi} r(1 + 2r \sin(\theta)) d\theta dr \\ &= \pi \end{aligned}$$

Exercise: Directly compute the line integrals...

NB = Often $\text{curl}(\vec{F})$ is simpler than \vec{F} .

EX: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and S the part of sphere $x^2 + y^2 + z^2 = 4$ inside cylinder $x^2 + y^2 = 1$ and above xy -plane.



Sol 1: (w/o Stokes's Theorem)

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)(1, 1, 0)$$

parameterize S (in cylindrical coordinates):

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, \sqrt{4-r^2} \rangle \text{ on } z^2 = 4-r^2 \quad (r, \theta) \in [0, 1] \times [0, 2\pi]$$

$$\vec{r}_r = \langle \cos(\theta), \sin(\theta), -\frac{1}{2}(4-r^2)^{-1/2} \cdot 2r \rangle = \langle \cos(\theta), \sin(\theta), -r(4-r^2)^{-1/2} \rangle$$

$$\vec{r}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$$

$$\begin{aligned} \vec{r}_r \times \vec{r}_\theta &= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -r(4-r^2)^{-1/2} \\ -r \sin(\theta) & r \cos(\theta) & 0 \end{bmatrix} \\ &= \langle r^2(4-r^2)^{-1/2} \cos(\theta), r^2(4-r^2)^{-1/2} \sin(\theta), r \rangle \end{aligned}$$

$$\text{curl}(\vec{F})(\vec{r}(r, \theta)) = (r \cos(\theta) - r \sin(\theta))(1, 1, 0)$$

$$\begin{aligned}
\therefore \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} &= \iint_D \text{curl}(\vec{F})(\vec{r}(r, \theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) \, dA \\
&= \iint_D (r \cos(\theta) - r \sin(\theta)) \langle 1, 1, 0 \rangle \cdot \\
&\quad \langle r^2(4-r^2)^{-1/2} \cos(\theta), r^2(4-r^2)^{-1/2} \sin(\theta), \\
&\quad r \rangle \, dA \\
&= 0
\end{aligned}$$

Sol 2: w/ Stokes' Theorem

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

Parameterize ∂S via $\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3} \rangle$

$$\vec{F}(\vec{r}(\theta)) = \langle \sqrt{3} \cos(\theta), \sqrt{3} \sin(\theta), \sin(\theta) \cos(\theta) \rangle$$

$$\vec{r}'(\theta) = \langle -\sin(\theta), \cos(\theta), 0 \rangle$$

$$\begin{aligned}
\therefore \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} &= \int_{\theta=0}^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta) \, d\theta \\
&= \int_{\theta=0}^{2\pi} (\sqrt{3} \cos(\theta), \sqrt{3} \sin(\theta), \sin(\theta) \cos(\theta)) \cdot \langle -\sin(\theta), \\
&\quad \cos(\theta), 0 \rangle \, d\theta = 0
\end{aligned}$$

NB: the "Stokes Equation" also implies

$$\begin{aligned}
\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} &= \int_T \text{curl}(\vec{F}) \cdot d\vec{S} \\
&\text{whenever } \partial S = \partial T \dots
\end{aligned}$$

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle x, y, z \rangle$ and C the boundary of the part of $z = 1 - x^2 - y^2$ in the first octant.

Sol: Note that C has "three pieces" (i.e. it is piecewise-defined)

Let's try Stokes's Theorem; $C = \partial S$ for S given by $\vec{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), 1 - r^2 \rangle$ on $(r, \theta) \in [0, 1] \times [0, \frac{\pi}{2}]$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = \langle y, z, x \rangle$$

$$\text{curl}(\vec{F})(\vec{r}(r, \theta)) = -\langle \sin(\theta), 1 - r^2, \cos(\theta) \rangle$$

$$\vec{r}_r = \langle \cos(\theta), \sin(\theta), -2r \rangle, \quad \vec{r}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r \sin(\theta) & r \cos(\theta) & 0 \end{bmatrix} = r \langle 2r \cos(\theta), 2r \sin(\theta) \rangle$$

$$\begin{aligned} \therefore \text{curl}(\vec{F})(\vec{r}(r, \theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) &= -r(2r^2 \sin(\theta) \cos(\theta) + 2(1 - r^2) \sin(\theta) + r \cos(\theta)) \\ &= -r^2(r \sin(2\theta) + 2(1 - r^2) \sin(\theta) + \cos(\theta)) \end{aligned}$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) d\vec{r} \\ &= \iint_D \text{curl}(\vec{F})(\vec{r}(r, \theta)) (\vec{r}_r \times \vec{r}_\theta) \\ &= \int_{r=0}^1 \int_{\theta=0}^{\pi/2} -r^2(r \sin(2\theta) + 2(1 - r^2) \sin(\theta) + \cos(\theta)) d\theta dr \\ &= \frac{2}{5} - \frac{1}{4} - 1 \end{aligned}$$